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A DOUBLE – SEQUENCE HYBRID S -ITERATION SCHEME FOR FIXED POINTS OF CONTINUOUS PSEUDOCONTRACTIONS

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ABSTRACT

*Let **C** be a bounded closed convex nonempty subset of a real Hilbert space **H**, It is proved that a double – sequence Hybrid S- Iteration scheme converges to a fixed point of continuous pseudocontractive map **T** which map **C** into **C**.*

Keywords : Continuous pseudocontraction, Double – sequence Hybrid S- Iteration, Hilbert space, Nonexpansive, Strong convergence.

2010 Mathematics Subject Classification: *47H09, 47H10, 47H17.*

I. INTRODUCTION

Let ***H*** be a a real Hilbert space and let ***C*** be a nonempty convex subset of a real Hilbert space ***H***.

Let ***T: C→C*** be a mapping.

The following Definitions have been studied widely and deeply by many authors; see, e.g., [1-21] for more details.

Definition 1.1. The mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Definition 1.2. The mapping T is said to be pseudocontractive if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

Definition 1.3. Let $\mathbb{N}$ denote the set of all the nonnegative integers and let E be a normed linear space. By a double sequence in E is meant by function $f : \mathbb{N} \times \mathbb{N} \to E$ defined by $f(n, m) = x_{n,m} \in E$. The double sequence $\{x_{n,m}\}$ is said to converges stongly to x^* if any $\varepsilon > 0$ there exist integer $N, M > 0$ such that $\forall\, n \geq N,\ m \geq M$, we have that $\|x_{n,m} - x^*\| < \varepsilon$. If $\forall\, n, r \geq N \text{ and } m, t \geq M$, we have that $\|x_{n,r} - x_{m,t}\| < \varepsilon$, then the double sequence is

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said to be Cauchy. Furthermore, if for each fixed n, $x_{n,m} \to x_n^*$ as $m \to \infty$ and then $x_n^* \to x^*$ as $n \to \infty$, then $x_{n,m} \to x^*$ as $n, m \to \infty$.

In the last few years or so, Several iteration process have been established for the constructive approximation of solution to several classes of (nonlinear) operator equations and several convergence results established using these iterative processes (see, e.g [1-21] and the reference cited therein). Most of these convergence results for the iterative solution of nonlinear operator equations or approximation of fixed points of nonlinear maps have used iteration process of the Mann , Ishikawa and Hybrid S- types.

The concept of Mann-type double –sequence iteration process introduced by C. Moore and he prove that it converges strongly to a fixed point of a continuous pseudocontraction which maps a bounded closed convex nonempty subset of a real Hilbert space into itself.

In this paper, we prove the same result for double –sequence of Hybrid S-iterative scheme.

II. MAIN RESULTS

Theorem 2.1. Let C be a bounded closed convex nonempty subset of a real Hilbert space H, and $T: C \to C$ be a continuous pseudocontractive map . Let $S: C \to C$ be nonexpansive mapping satisfying that $\|x - Sy\| \leq \|Sx - Sy\|$ for all $x, y \in C$ and $\{\beta_n\}_{n\geq 0}, \{\alpha_k\}_{k\geq 0} \subset (0,1)$ be real sequence satisfying the following condition :

1) $\lim_{k\to\infty} \alpha_k = 1$(monotonically);
2) $\lim_{k,j\to\infty} \left(\frac{\alpha_k - \alpha_j}{1-\alpha_k}\right) = 0$, for all $0 < j \leq k$;
3) $\lim_{n\to\infty} \beta_n = 0$;
4) $\sum_{n\geq 0} \beta_n = \infty$.

For an arbitrary but fixed $\mu \in C$, and for each $k \geq 0$, define $T_k: C \to C$ by $T_k x = (1-\alpha_k)\mu + \alpha_k T x, \forall x \in C$.

Then, the double sequence $\{x_{k,n}\}_{k\geq 0, n\geq 0}$ generated from an arbitrary $x_{0,0} \in C$ by

$$x_{k,n+1} = Sy_{k,n}$$
$$y_{k,n} = (1-\beta_n)x_{k,n} + \beta_n T_k x_{k,n}, \quad k, n \geq 0.$$

Converges strongly to a fixed point x_∞^* of T in C.

Proof: Consider

$\langle T_k x - T_k y, x - y\rangle = \alpha_k \langle Tx - Ty, x - y\rangle \leq \alpha_k \|x - y\|^2$

So that $\forall k \geq 0$, T_k is continuous and strongly pseudocontractive. Also, C is invariant under T_k, $\forall k$ by convexity.

Hence, T_k has a unique fixed point $x_k^* \in C, \forall k \geq 0$. It thus suffices to prove the following:

i. For each fixed $\forall k \geq 0$, $x_{k,n} \to x_k^* \in C$ as $n \to \infty$;

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ii. $x_k^* \to x_\infty^* \in C$ as k$\to \infty$;

iii. $x_\infty^* \in F(T)$.

Now, Let $b = diam\ C$ and $\gamma_k = (1-\alpha_k) \in (0,1), \forall\ k.$ Then

$$\begin{aligned}
\|x_{k,n+1} - x_k^*\|^2 = \|Sy_{k,n} - x_k^*\|^2 &\leq \|Sy_{k,n} - x_k^*\|^2 \\
&= \|y_{k,n} - x_k^*\|^2 \\
&= \|(1-\beta_n)x_{k,n} + \beta_n T_k x_{k,n} - x_k^*\|^2 \\
&= \|x_{k,n} - x_k^* - \beta_n (x_{k,n} - T_k x_{k,n})\|^2 \\
&= \|x_{k,n} - x_k^*\|^2 - 2\beta_n \langle x_{k,n} - T_k x_{k,n}, x_{k,n} - x_k^* \rangle + {\beta_n}^2 \|x_{k,n} - T_k x_{k,n}\|^2 \\
&= \|x_{k,n} - x_k^*\|^2 - 2\beta_n (1-\alpha_k)\|x_{k,n} - x_k^*\|^2 + b^2 {\beta_n}^2 \\
&= (1 - 2\gamma_k \beta_n)\|x_{k,n} - x_k^*\|^2 + b^2 {\beta_n}^2
\end{aligned}$$

If we set

$$\theta_{k,n} = \|x_{k,n} - x_k^*\|, \quad \delta_{k,n} = 2\gamma_k \beta_n, \quad \sigma_{k,n} = b^2 {\beta_n}^2$$

Then we have

$$\theta_{k,n+1}^2 \leq (1 - \delta_{k,n})\theta_{k,n}^2 + \sigma_{k,n},$$

So that observing that

$$\sigma_{k,n} = o(\delta_{k,n}), \quad \lim_{n\to\infty} \delta_{k,n} = 0 \quad \text{and} \quad \sum_{n\geq 0} \sigma_{k,,n} = 0$$

We then have that $\theta_{k,n} \to 0$ as $n \to \infty$. So the part (i) is proved.

Now,

$$\begin{aligned}
\|x_k^* - Tx_k^*\| = \|x_k^* - {\alpha_k}^{-1}x_k^* - {\alpha_k}^{-1}(1-\alpha_k)\mu\| \\
&= \left\|\left(\frac{\alpha_{k-1}}{\alpha_k}\right)x_k^* - \left(\frac{1-\alpha_k}{\alpha_k}\right)\mu\right\| \\
&= \left(\frac{1-\alpha_k}{\alpha_k}\right)(\|x_k^*\| + \|\mu\|) \\
&\leq 2b\left(\frac{1-\alpha_k}{\alpha_k}\right)
\end{aligned}$$

So that

$$\lim_{k\to\infty} \|x_k^* - Tx_k^*\| = 0$$

And hence , $\{x_k^*\}$ is an approximate fixed point sequence for T. Also, supposing that x_∞^* is a fixed point of T, then

$$\lim_{k\to\infty} \|x_\infty^* - T_k x_\infty^*\| = \lim_{k\to\infty} (1-\alpha_k)\|x_\infty^* - \mu\| \leq 2b \lim_{k\to\infty} (1-\alpha_k) = 0$$

Now, for all $0 < j \leq k$,

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$$\begin{aligned}\| x_k^* - x_j^* \|^2 &= \langle x_k^* - x_j^*, x_k^* - x_j^* \rangle \\ &= \langle T_k x_k^* - T_k x_j^*, x_k^* - x_j^* \rangle \\ &= (\alpha_j - \alpha_k)\langle \mu, x_k^* - x_j^* \rangle + \langle \alpha_k T x_k^* - \alpha_j T x_j^*, x_k^* - x_j^* \rangle \\ &= (\alpha_j - \alpha_k)\langle \mu, x_k^* - x_j^* \rangle + (\alpha_k - \alpha_j)\langle T x_j^*, x_k^* - x_j^* \rangle + \alpha_k \langle T x_k^* - T x_j^*, x_k^* - x_j^* \rangle \\ &\leq |\alpha_k - \alpha_j| \| x_k^* - x_j^* \| (\| T x_j^* \| + \| \mu \|) + \alpha_k \| x_k^* - x_j^* \|^2\end{aligned}$$

So that

$$(1 - \alpha_k)\| x_k^* - x_j^* \|^2 \leq |\alpha_k - \alpha_j| \| x_k^* - x_j^* \| (\| T x_j^* \| + \| \mu \|)$$

And hence,

$$\lim_{k,j \to \infty} \| x_k^* - x_j^* \| \leq 2b \lim_{k,j \to \infty} \left(\frac{\alpha_k - \alpha_j}{1 - \alpha_k} \right) = 0$$

Thus, $\{x_k^*\}$ is a Cauchy sequence, and hence there exists $x_\infty^* \in C$ such that $x_k^* \to x_\infty^*$ as $k \to \infty$.

So the part (ii) is proved.

By continuity, $T x_k^* \to T x_\infty^*$ as $k \to \infty$. But $x_k^* - T x_k^* \to 0$ as $k \to \infty$. Hence, $x_\infty^* \in F(T)$.

This completes the proof.

Corollary 2.2: Let C be a bounded closed convex nonempty subset of a real Hilbert space H, and $T: C \to C$ be a continuous pseudocontractive map . Let $S: C \to C$ be nonexpansive mapping satisfying that $\|x - Sy\| \leq \|Sx - Sy\|$ for all $x, y \in C$ and $\{\beta_n\}_{n \geq 0}, \{\alpha_k\}_{k \geq 0} \subset (0,1)$ be real sequence satisfying the following condition :

1) $\lim_{k \to \infty} \alpha_k = 1$(monotonically);
2) $\lim_{k,j \to \infty} \left(\frac{\alpha_k - \alpha_j}{1 - \alpha_k} \right) = 0$, for all $0 < j \leq k$;
3) $\lim_{n \to \infty} \beta_n = 0$;
4) $\sum_{n \geq 0} \beta_n = \infty$.

For an arbitrary but fixed $\mu \in C$, and for each $k \geq 0$, define $T_k: C \to C$ by $T_k x = \alpha_k T x, \forall x \in C$. Then, the double sequence $\{x_{k,n}\}_{k \geq 0, n \geq 0}$ generated from an arbitrary $x_{0,0} \in C$ by

$$x_{k,n+1} = S y_{k,n}$$
$$y_{k,n} = (1 - \beta_n) x_{k,n} + \beta_n T_k x_{k,n}, \quad k, n \geq 0.$$

Converges strongly to a fixed point x_∞^* of T in C.

Proof: This follows from Theorem 2.1 on setting $\mu = 0 \in C$.

Corollary 2.3: In Theorem 2.1, Let T be a non expansive map. Then the same conclusion is obtained.

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Proof: Observe that every non expansive map is continuous pseudocontraction.

Remark 2.5: If we put $S = I$ in Theorem 2.1 then we get the result of C. Moore (see [17])

Remark 2.4: Prototypes of the sequence $\{\alpha_k\}$ are: $\forall\, k \ \geq 0,$

$$\alpha_k = exp\left\{-\frac{1}{k+1}\right\}, \quad \alpha_k = \log_e\left(e - \frac{1}{k+1}\right), \quad \alpha_k = 1 - \frac{1}{\log_m(k+m+1)}, \quad m > 1.$$

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