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Fixed point theorems in fuzzy metric space via semi-compatible and occasionally weakly compatible mappings

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ABSTRACT

In this paper, the concept of semi-compatibility and occasionally weakly compatibility in fuzzy metric space has been applied to prove common fixed point theorems for four self maps

Keyword : Occasionally weakly compatible mapping,semi compatible mapping,fuzzy metric space.

I .INTRODUCTION

Fuzzy set was defined by Zadeh [18]. Kramosil and Michalek [10] introduced fuzzy metric space, George and Veermani [7] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Many researchers have obtained common fixed point theorem for mappings satisfying different types of commutativity conditions. Vasuki [17] proved fixed point theorems for R-weakly commuting mappings. Pant [12, 13, 14] introduced the new concept of reciprocally continuous mappings and established some common fixed point theorems. Balasubramaniam et al. [6] have shown that Rhoades [15] open problem on the existence of contractive definition which generates a fixed point but does not force the mappings to be continuous at the fixed point, posses an affirmative answer. Pant and Jha [14] obtained some anologous results proved by Balasubramanium et al.[6]. Recent literature on fixed point in fuzzy metric space can be viewed in [1, 2, 3, 4,5, 8, 11, 16].

II. PRELIMINARIES

Definition 2.1 A binary operation * : [0,1] * [0,1]→[0,1] is called a t-norm if ([0,1],*) is an abelian topological monoid with unit 1 such that $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for $a,b,c,d \in [0,1]$.

Examples of t-norms are a*b=ab and a*b = min {a,b}.

Definition 2.2 The 3-tuple (X, M, *) is said to be a Fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions : for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) M(x, y, 0) = 0,

(FM-2) M(x, y, t) =1 for all t > 0 if and only if x = y,

(FM-3) M (x, y, t) = M (y, x, t),

(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$

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Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1 Let (X, d) be a metric space. Define a * b = min {a, b} and

M(x,y,t) = t/t+d(x,y) for all x, y ∈ X and all t > 0. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3 A sequence $\{x_n\}$ in a Fuzzy metric space (X,M,*) is said to be a Cauchy sequence if and only if for each ε >0, t > 0,there exists $n_{0} \in$ N such that M (x_n,x_m,t) > 1- ε for all n,m ≥ $n_{0.}$

The sequence $\{x_n\}$ is said to converge to a point x in X if and only if for each ε > 0, t > 0 there exit n_0 ∈N such that M (x_n,x_m,t) > 1- ε, for all n ≥ $n_{0.}$

A Fuzzy metric space (X,M,*) is said to be complete if every cauchy sequence in it converges to a point it.

Definition 2.4 Self mappings A and S of a Fuzzy metric space (X,M,*) are said to be compatible if and only if $M(ASx_{n,}SAx_{n,}t)\rightarrow 1$ for all t > 0 whenever $\{x_n\}$ is a sequence in X such that $Sx_{n,}Ax_n\rightarrow p$ for some p in X as $n\rightarrow\infty$.

Definition 2.5 Suppose A and S be two maps from a Fuzzy metric space (X,M,*) into itself.Then they are said to be semicompatible if limASx_n=Sx whenever $\{x_n\}$ is a sequence such that limAx_n = limSx_n= x ∈ X.

It follows that (A,S) is semi compatible and Ay = Sy imply ASy=SAy by taking $\{x_n\}$=y and x = Ay = Sy.

Definition 2.6 Self maps A and S of a Fuzzy metric space (X,M,*) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points if Ap = Sp for some p ∈ X then ASp = SAp.

Definition 2.7 Two self maps f and g of a set X are occasionally weakly compatible(owc) iff there is a point x in Xwhich is a coincidence point of f and g at which f and g commute.

Lemma 2.1 Let (X,M,*) be a Fuzzy metric space.Then for all x,y ∈X,M (x,y,.)is a non decreasing function.

Lemma 2.2 Let (X,M,*) be a fuzzy metric space.If there exists k ∈ (0,1) such that for all x,y ∈ X .M(x,y,kt) ≥ M(x,y,t) for all t > 0,then x = y.

Lemma 2.3 [14] Let X ba a set ,f,g owc self maps X.If f and g have a unique point of coincidence,w = fx = gx,then w is the unique common fixed point of f and g.

Example 2.2 Let X = [0, 2] and a *b = min {a, b}. Let M(x,y,t) = t /t + d(x,y)

be the standard Fuzzy metric space induced by d, where d(x, y) = | x − y| for all x, y ∈ X, define

A(x) = {2, x ∈ [0,1] S(x) = 1, x ∈ [0,1)

2, x = 1

{ x/2 , x ∈ (1,2] x+3/5 x ∈ (1,2]

Now we have S(1) = 2 = A(1), and S(2) = 1 = A(2)

also SA(1) = AS(1) and AS(2) = 2 = AS(2)

Let x_n = 2- 1/2n

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Hence $Ax_n \to 1$, $Sx_n \to 1$ and $ASx_n \to 2$

Therefore $M(ASx_n, Sy, t) = (2, 2, t) = 1$.

Hence (A, S) is Semi compatible.

Example 2.3 Let X = [0,1] and d be the usual metric on X.Define f,g:X→X by

$f(x) = 2x$ for all $x \in [0,1]$

$g(x) = 0, x \in [0,1]$

$= 1, x=0$

Taking $x_n = 1/n$

Since $\lim fx_n = \lim gx_n = 0$

Also $\lim fgx_n = \lim f(0) = 0$ (1)

$\lim gfx_n = \lim g(2x_n) = \lim 2g(x_n) = \lim g(1/n) = 0$ (2)

equation (1) and (2) shows that maps f and g are semi compatible.

Now,

$f(1) = 2, g(1) = 0$

$f(0) = 0, g(0) = 1$

$f(g(1)) = f(0) = 0, f(g(0)) = =f(1) = 2$

$g(f(1)) = g(2) = 0, g(f(0)) = g(0) = 1$

$f(g(1)) = g(f(1)), f(g(0)) \neq g(f(0))$

Hence semi compatible does not implies occasionally weakly compatible.

Note : Above examples shows that owc impiles semi compatible but converse is not true.

III. MAIN RESULTS

Theorem 1. Let (X,M,*) be a complete fuzzy metric space and A,B,S,T be a self mappings of X .Let {A.S} be semi compatible and {B,T} be owc.if there exists $q \in (0,1)$ such that

$M(Ax,By,qt) \geq \alpha_1 M(Sx,Ty,t) + \alpha_2 M(Ax,Ty,t) + \alpha_3 M(By,Sx,t)$ (1)

For all $x,y \in X$,where $\alpha_1, \alpha_2, \alpha_3 > 0$ and $(\alpha_1 + \alpha_2 + \alpha_3) > 1$ then there exists a unique point $w \in X$ such that Aw = Sw = w and unique point $z \in X$ such that Bz = Tz = z.Moreover,z = w,so that there is a common fixed point of A,B,S and T.

Proof : Let {A.S} be semi compatible and {B,T} be owc,so there is a point $x,y \in X$ such that Ax = Sx implies ASx = SAx and $\lim Sx_n = \lim Tx_n = x \in X$.and By = Ty.

Claim : Ax = By.If not,by inequality (1)

$M(Ax,By,qt) \geq \alpha_1 M(Sx,Ty,t) + \alpha_2 M(Ax,Ty,t) + \alpha_3 M(By,Sx,t)$

$= \alpha_1 M(Ax,By,t) + \alpha_2 M(Ax,By,t) + \alpha_3 M(By,Ax,t)$

$= (\alpha_1 + \alpha_2 + \alpha_3) M(Ax,By,t)$

A contradiction ,Since $(\alpha_1 + \alpha_2 + \alpha_3) > 1$.Therefore Ax = By,i.e. Ax = Sx = By = Ty.

Since (X,M,*) is complete,{yn} converges to some point $z \in X$.Also its subsequences converges to the sme point i.e.$z \in X$.

i.e. $Bx_{2n+1} \to z$ and $Tx_{2n+1} \to z$ (2)

$Ax_{2n} \to z$ and $Sx_{2n} \to z$ (3)

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Since (A,S) is semi compatible pair,we have

$A(S)\, x_{2n} \to Az$

$\&A(S)\, x_{2n} \to Sz$

Since the limit in Fuzzy metric space is unique,we get

$Az = Sz$ (4)

Putting $x = Ax_{2n}$ & $y = x_{2n+1}$ in (1),we have,

$M(AAx_{2n}, Bx_{2n+1}, qt) \geq \alpha_1 M(SAx_{2n}, Tx_{2n+1}, t) + \alpha_2 M(AAx_{2n}, Tx_{2n+1}, t) + \alpha_3 M(Bx_{2n+1}, SAx_{2n}, t)$

Taking $n \to \infty$ & using (2),(3) & (4)

$M(Az,z,qt) \geq \alpha_1 M(Az,z,t) + \alpha_2 M(Az,z,t) + \alpha_3 M(z,Az,t)$

$M(Az,z,qt) \geq (\alpha_1 + \alpha_2 + \alpha_3) M(Az,z,t)$

Therefore

$z = Az$, Since $(\alpha_1 + \alpha_2 + \alpha_3) > 1$

$Az = z = Sz = By = Ty$

So $Ax = Az$ and $w = Ax = Sx$ is unique point of coincidence of A and S.By lemma 2.3 w is the only common fixed point of A and S i.e.,$w = Aw = Sw$.Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Uniqueness: Let w be another common fixed point of A,B,S & T.

Then $Aw = Bw = Sw = Tw = w$

Put $x = z$ & $y = w$ in (1) ,we get

$M(Az, Bw, qt) \geq \alpha_1 M(Sz, Tw, t) + \alpha_2 M(Az, Tw, t) + \alpha_3 M(Bw, Sz, t)$

$M(z,w,qt) \geq \alpha_1 M(z,w,t) + \alpha_2 M(z,w,t) + \alpha_3 M(w,z,t)$

$M(z,w,qt) \geq (\alpha_1 + \alpha_2 + \alpha_3) M(z,w,t)$

Therefore ,$z = w$ Since $(\alpha_1 + \alpha_2 + \alpha_3) > 1$

Therefore z is the unique common fixed point of self maps A,B,S and T.

Theorem 2 . Let $(X,M,*)$ be a complete fuzzy metric space and let A and S be selfmappings of X.Let the A and S are owc.If there exists $q \in (0,1)$ for all $x,y \in X$ and $t > 0$

$M(Sx,Sy,qt) \geq \alpha_1 M(Az,Ay,t) + \alpha_2 M(Sx,Ay,t) + \alpha_3 M(Sy,Ax,t) + \alpha_4 M(Ax,Sy,t)$ (5)

For all $x,y \in X$,where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$ and $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$.Then A and S have a unique common fixed point.

Proof : Let the pair {A,S} be owc,So there exists a points $x \in X$ such that $Ax = Sx$.Suppose that there exists another point $y \in X$ for which $Ay = Sy$.

Claim : $Sx = Sy$.If not,by inequality (5)

$M(Sx,Sy,qt) \geq \alpha_1 M(Ax,Ay,t) + \alpha_2 M(Sx,Ay,t) + \alpha_3 M(Sy,Ax,t) + \alpha_4 M(Ax,Sy,t)$

$= \alpha_1 M(Sx,Sy,t) + \alpha_2 M(Sx,Sy,t) + \alpha_3 M(Sy,Sx,t) + \alpha_4 M(Sx,Sy,t)$

$= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) M(Sx,Sy,t)$

A contradiction,Since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$.Therefore $Sx = Sy$.Therefore $Ax = Ay$ and Ax is unique.From Lemma 2.3,A and S have a unique fixed point.

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IV. CONCLUSION

In this paper, the concept of semi-compatibility and occasionally weakly compatibility in fuzzy metric space has been applied to prove common fixed point theorems for four self maps. Our result generalizes the result of Aage et. al.[1] published in Int. J. Open Problems Compt. Math., 3(2), (2010), 123-131.

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