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Intuitionistic Fuzzy Number and Their Geometrical Analysis

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Abstract:
Intuitionistic fuzzy numbers are geometrically significant because they are a more flexible representation of ambiguity and uncertainty than conventional fuzzy numbers. When modeling real-world issues, where exact values aren't always accessible, Intuitionistic fuzzy numbers are a better fit since they include a broader range of values. This representational flexibility may aid decision-making by better reflecting the actual nature of ambiguity in each particular context. Additionally, Intuitionistic fuzzy numbers may represent the idea of uncertainty or reluctance to give a single value, which is prevalent in many real-life situations. By considering this factor, decision-makers may have a better grasp of the degree of uncertainty and make smarter decisions. Intuitionistic fuzzy numbers are important because they improve decision-making in domains like engineering, economics, and finance by offering a more realistic and subtle way to describe uncertainty.

Keyword: *Intuitionistic fuzzy Number, Geometrical Significance, Algebraic Analysis.*

1. Introduction:

When modeling imperfect measurements or unknown characteristics in a system design, Intuitionistic fuzzy numbers may be useful in engineering. Engineers may better evaluate the robustness of their systems and account for any variances in performance by evaluating both the degree of membership and non-membership. Analogously, Intuitionistic fuzzy numbers may aid economists and financiers in more precisely assessing the return and risk profiles of investment portfolios by factoring in not only the anticipated results but also the degree of certainty or uncertainty linked to each possibility. Taking a comprehensive view of managing uncertainty may help design strategies that are better equipped to withstand and react to changing market circumstances. Nevertheless, decision-making processes might become more complicated and subjective if Intuitionistic fuzzy numbers are relied upon exclusively, which could cause confusion or misunderstanding of outcomes. Furthermore, it is possible that these areas still have a stronger affinity for and familiarity with more conventional statistical approaches and probabilistic models.

1.1 Intuitionistic Fuzzy Numbers

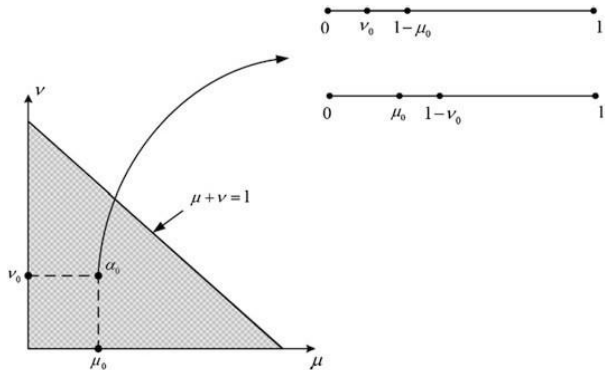

Fig. 1.1 Expressing α_0 as the subintervals of [0,1]

The above-mentioned two representations of an IFN play important roles in this study . We will show some theorems and conclusions about IFNs from two different angles

1.2 Basic Operations Between Intuitionistic Fuzzy Numbers

Fundamental Operations Involving Intuitionistic Fuzzy Numbers: The fundamental operations with Intuitionistic fuzzy numbers include addition, subtraction, multiplication, and division. The summation of two Intuitionistic fuzzy numbers entails the distinct addition of their membership and non-membership values. Subtraction involves deducting the membership and non-membership values of the second number from those of the first number. Multiplication and division are intricate procedures requiring precise formulae for proper result computation. For instance, given two Intuitionistic fuzzy numbers A = (0.4, 0.3, 0.2) and B = (0.6, 0.2, 0.1), their addition yields (1.0, 0.5, 0.3), where the membership value is the aggregate of both numbers' membership values, and the non-membership value is the total of their non-membership values. The subtraction yields (0.2, 0.1, 0.1), where the membership value is derived from the difference between the first and second number's membership values, and the non-membership value is obtained from the difference between their non-membership values. The multiplication and division of Intuitionistic fuzzy numbers adhere to similar principles, but with more complex computations required to ascertain the final values appropriately. These activities are crucial for managing uncertainty and ambiguity in decision-making processes.

The real numbers and complex numbers possess distinct operational rules; similarly, the IFNs have certain operations that will be elucidated in this section. XU and Yager (2006, 2007) established the addition and multiplication operations for any two IFNs based on the addition and multiplication of A-IFSs as follows:

Definition 1.3 (Xu and Yager 2006, 2007). Let $\alpha = (\mu_\alpha, \upsilon_\alpha)$ and $\beta = (\mu_\beta, \upsilon_\beta)$ represent be two IFNs. The operations of addition and multiplication between them are defined as follows:

(Addition) $\alpha \oplus \beta = (\mu_\alpha + \mu_\beta - \mu_\alpha\, \mu_\beta, \upsilon_\alpha \upsilon_\beta)$;

(Multiplication) $\alpha \otimes \beta = (\mu_\alpha \mu_\beta, \upsilon_\alpha + \upsilon_\beta - \upsilon_\alpha \upsilon_\beta)$.

According to the addition and multiplication operations of IFNs, we can easily get that $\alpha \oplus \alpha = (1 - (1 - \mu_\alpha)^2, \upsilon_\alpha^2)$,

$\alpha \oplus \alpha \oplus \alpha = (1-(1-\mu_\alpha)^3, v_\alpha^3)$,
$\alpha \otimes \alpha = (\mu_\alpha^2, 1-(1-v_\alpha)^2)$,
$\alpha \otimes \alpha \otimes \alpha = (\mu_\alpha^3, 1-(1-v_\alpha)^3)$ and so on.

Therefore, it is appropriate to provide the following definition of scalar multiplication and power operation of IFNs:

Definition 1.4 (Xu and Yager 2006, 2007) Let $\alpha = (\mu_\alpha, v_\alpha)$ be an IFN, and the parameter λ be a real number meeting $\lambda > 0$. Then we have

(Scalar-multiplication) $\lambda\alpha = (1-(1-\mu_\alpha)^\lambda, v_\alpha^\lambda)$;

(Power operation) $\alpha^\lambda = (\mu_\alpha^\lambda, 1-(1-v_\alpha)^\lambda)$.

In order to understand these operations better, we firstly transform $\alpha = (\mu_\alpha, v_\alpha)$ and $\beta = (\mu_\beta, v_\beta)$ into $[v_\alpha, 1-\mu_\alpha]$ and $[v_\beta, 1-\mu_\beta]$ respectively. Then there are the following processes:

$$(\mu_\alpha, v_\alpha) \oplus (\mu_\beta, v_\beta) = (1-(1-\mu_\alpha)(1-\mu_\beta), v_\alpha v_\beta)$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

$$[v_\alpha, 1-\mu_\alpha] \oplus [v_\beta, 1-\mu_\beta] = [v_\alpha v_\beta, (1-\mu_\alpha)(1-\mu_\beta)]$$

Consequently, the incorporation of IFNs effectively amplifies the upper limit $1-\mu_\alpha$ and the lower bound v_α of $[v_\alpha, 1-\mu_\alpha]$ by the upper bound $1-\mu_\beta$ and the lower bound v_β of $[v_\beta, 1-\mu_\beta]$ to an interval $[v_\alpha v_\beta$, $(1-\mu_\alpha)(1-\mu_\beta)]$.In addition, we can also get the scalar-multiplication $\lambda\alpha$ of IFNs by dealing with the upper and lower bounds of $[v_\alpha, 1-\mu_\alpha]$ and $[v_\beta, 1-\mu_\beta]$,respectively. On the other hand, if we transform α and β into $[\mu_\alpha, 1-v_\alpha]$
and $[\mu_\beta, 1-v_\beta]$ we can analyze the multiplication and power operations of IFNs in the same way. The processes can be shown in Fig. 1.3 (Lei und Xu 2015b).

Utilising the operations of addition and multiplication among IFNs, we can delineate their inverse operations (subtraction and division) as follows:

Definition 1.5 (Lei and Xu 2015b) Let $\alpha = (\mu_\alpha, v_\alpha)$ and $\beta = (\mu_\beta, v_\beta)$ be two IFNs.

Then we get

(Subtraction) $\beta \ominus \alpha = \begin{cases} \left(\frac{\mu_\beta-\mu_\alpha}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right) & if\ 0 \le \frac{v_\beta}{v_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1; \\ o, & otherwise. \end{cases}$

where O is the IFN (0,1).

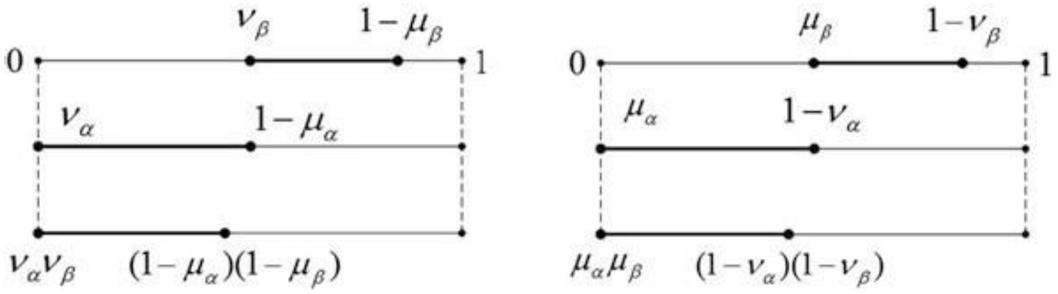

Fig. 1.3 Addition and multiplication between α and β

(Division) $\boldsymbol{\beta} \oslash \boldsymbol{\alpha} = \begin{cases} \left(\frac{\mu_\beta}{\mu_\alpha}, \frac{v_\beta - v_\alpha}{1-v_\alpha}\right) & if\ 0 \le \frac{\mu_\beta}{\mu_\alpha} \le \frac{1-v_\beta}{1-v_\alpha} \le 1; \\ E, & otherwise. \end{cases}$

where $\boldsymbol{E}$ is actually (1,0),

Obviously, the subtraction and the division defined in Definition 1.5 are the inverse operations of addition and multiplication of IFNs, respectively. It means that there are $(\boldsymbol{\alpha} \oplus \boldsymbol{\beta}) \ominus \boldsymbol{\alpha} = \boldsymbol{\beta}$, $(\boldsymbol{\alpha} \oplus \boldsymbol{\beta}) \ominus \boldsymbol{\beta} = \boldsymbol{\alpha}$, $(\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \oslash \boldsymbol{\alpha} = \boldsymbol{\beta}$ and $(\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \oslash \boldsymbol{\beta} = \boldsymbol{\alpha}$. In addition, we can calculate the difference between $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ by using the following formula:

$$\boldsymbol{\beta} \ominus \boldsymbol{\alpha} = \left(\frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right)$$

if only β and α satisfy that $0 \le \frac{v_\beta}{v_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$. However, we notice that the result of $\left(\frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right)$ may not be an IFN, which means that at least one of three inequalities $0 \le \frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha} \le 1, 0 \le \frac{v_\beta}{v_\alpha} \le 1$ and $0 \le \frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha} + \frac{v_\beta}{v_\alpha} \le 1$ does not hold, if $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ do not meet $0 \le \frac{v_\beta}{v_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$. Meanwhile, in order to let the subtraction operation of IFNs have the closure, Definition 1.5 defines the difference $\boldsymbol{\beta} \ominus \boldsymbol{\alpha} = \boldsymbol{O}$ when the condition $0 \le \frac{v_\beta}{v_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$ does not hold, in order that the subtraction operation of IFNs has the closure. However, in this case, the result O of $\boldsymbol{\beta} \ominus \boldsymbol{\alpha} = \boldsymbol{O}$ is almost meaningless because the difference result completely loses the information of minuend and subtrahend ($\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$).

1.2.1 Geometrical Analysis of the Operations of IFNs

For every two specified real values y and z, there exists a real number x such that $y = x * z$ where the operator "*" represents one of the four arithmetic operations: addition, subtraction, multiplication, or division of real numbers. Inspired by this, this subsection will examine whether a similar conclusion can be drawn for complex numbers, specifically whether there exists an IFN β such that $\boldsymbol{\alpha} = \boldsymbol{\alpha_0} * \boldsymbol{\beta}$, where "*" represents one of the operations of IFNs: "⊕", "⊖", "⊗", and "⊘". The conclusion applicable to real numbers will also be assessed for complex numbers. Subsequently, we shall provide analytical data for further elucidation. Initially, the following findings are presented by Lei and Xu :

(1) In Fig. 1.4 (Lei and Xu 2016a), for any IFN $\boldsymbol{\beta} = (\mu_\beta, v_\beta)$ in the area $S_\oplus(\alpha)$, it must satisfy the condition $0 \le \frac{v_\beta}{v_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$. Hence, $\boldsymbol{\beta}|\ominus|\ \boldsymbol{\alpha} = \left(\frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right)$ must be an IFN. If we let $x = \left(\frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right)$, then there exists x

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meeting β = $\alpha \oplus x$. In addition, for a set $\{\boldsymbol{\alpha} \oplus \boldsymbol{x}|\boldsymbol{x} \in \blacktriangle\}$, then $\boldsymbol{\beta} \in \{\boldsymbol{\alpha} \oplus \boldsymbol{x}|\boldsymbol{x} \in \blacktriangle\}$. Hence, $S_{\oplus}(\boldsymbol{\alpha}) \subseteq \{\boldsymbol{\alpha} \oplus \boldsymbol{x}|\boldsymbol{x} \in \blacktriangle\}$.On the other hand, if only the IFN $\boldsymbol{\beta}$ belongs to the set

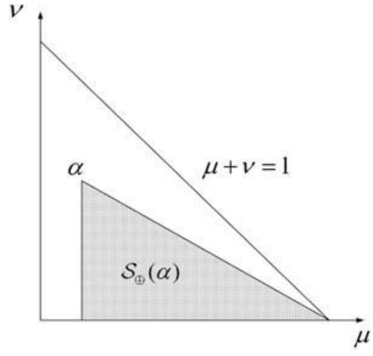

Fig. 1.4 Addition region $S_{\oplus}(\alpha)$ of α

$\{\boldsymbol{\alpha} \oplus \boldsymbol{x}|\boldsymbol{x} \in \blacktriangle\}$, then it must satisfy $0 < \frac{v_\beta}{v_\alpha} \leq \frac{1-\mu_\beta}{1-\mu_\alpha} \leq 1$ and fall into the area $S_{\oplus}(\boldsymbol{\alpha})$ .So we also have the conclusion that $\{\boldsymbol{\alpha} \oplus \boldsymbol{x}|\boldsymbol{x} \in \blacktriangle\} \subseteq S_{\oplus}(\boldsymbol{\alpha})$, and thus, $S_{\oplus}(\boldsymbol{\alpha}) = \{\boldsymbol{\alpha} \oplus \boldsymbol{x}|\boldsymbol{x} \in \blacktriangle\}$. We call $S_{\oplus}(\boldsymbol{\alpha})$ the addition region of $\boldsymbol{\alpha}$, which con tains the following two meanings:

(a) Any $\boldsymbol{\alpha} \oplus \boldsymbol{x}$ $(\boldsymbol{x} \in \blacktriangle)$ must fall into the area $S_{\oplus}(\boldsymbol{\alpha})$;

(b) For any $\beta \in S_{\oplus}(\boldsymbol{\alpha})$, $\beta \ominus \boldsymbol{\alpha} = \left(\frac{\mu_\beta-\mu_\alpha}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right)$ is still an IFN.

According to the definition of $S_{\oplus}(\boldsymbol{\alpha})$, we have the corresponding notion of the subtraction region $S_{\ominus}(\boldsymbol{\alpha})$, which can be expressed as follows:

(2) If we let the set $S_{\ominus}(\boldsymbol{\alpha})$, be $\{\boldsymbol{\alpha} \ominus \boldsymbol{x}|\boldsymbol{x} \in \blacktriangle\}$, then there must exist an IFN x_0 such that $\beta \oplus x_0 = \boldsymbol{\alpha}$ for any given $\beta \in \{\boldsymbol{\alpha} |\ominus|\boldsymbol{x}|\boldsymbol{x} \in \blacktriangle\}$. Hence, we have $\boldsymbol{\alpha} \in$ $S_{\oplus}(\beta)$ based on the definition of addition regions. Therefore, the equation $S_{\ominus}(\boldsymbol{\alpha}) = \{\boldsymbol{\alpha} |\ominus|\boldsymbol{x}|\boldsymbol{x} \in \blacktriangle\} = \{\beta|\alpha \in S_{\oplus}(\beta)\}$ holds, which successfully associates the notion of subtraction regions with addition regions defined by (1) aforementioned. According to $S_{\ominus}(\boldsymbol{\alpha}) = \{\beta|\alpha \in S_{\oplus}(\beta)\}$, we can get that the area of $S_{\ominus}(\boldsymbol{\alpha})$ is just the shadow region of Fig. 1.5 (Lei and Xu 2016a), because there

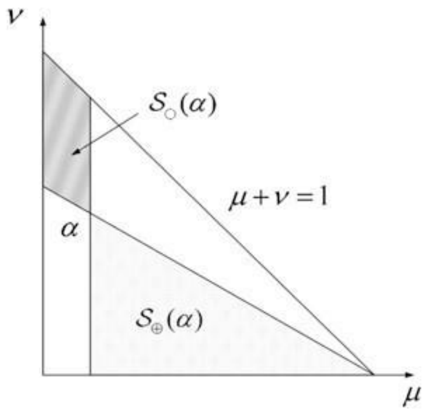

Fig .1.5 Subtraction region $S_{\ominus}(\boldsymbol{\alpha})$ of $\boldsymbol{\alpha}$

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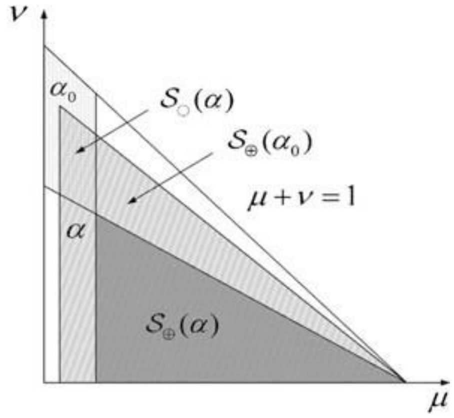

Fig. 1.6 Addition region of α_0 in a subtraction region

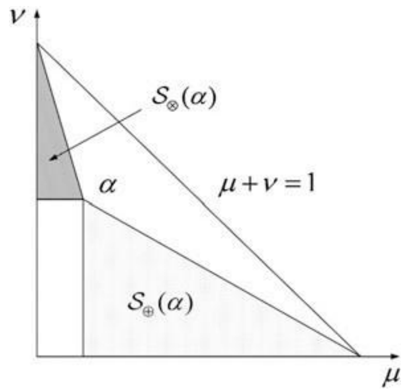

Fig. 1.7 Multiplication region $S_{\otimes}$ $(\boldsymbol{\alpha})$ of $\boldsymbol{\alpha}$

must be $\boldsymbol{\alpha} \in S_{\oplus}$ (α_0) for any α_0 in the shadow region of Fig. 1.6 (Lei and Xu 2016a). Meanwhile, we can also have a conclusion that $S_{\oplus}$ $(\boldsymbol{\alpha}) \subseteq S_{\oplus}$ $(\boldsymbol{\alpha}_0)$if only $\boldsymbol{\alpha} \in \boldsymbol{S}_{\oplus}$ $(\boldsymbol{\alpha_0})$

Subsequently, we will examine the multiplication region $S_{\otimes}$ (α) and the division region $S_{\oslash}(\alpha)$ of IFNs in a similar manner (Lei and Xu 2015b, 2016a).

(3) The multiplication region $S_{\otimes}$ (α) corresponds to the shadow region depicted in Fig. 1.7, as per the operations of multiplication and division of IFNs.

(4) In a manner analogous to the method that establishes the subtraction region from the addition region in (2), the division region can be delineated in relation to the multiplication region in (3). Any IFN $\boldsymbol{\alpha_0}$ in the shadow region of Fig. 1.8 must satisfy $\alpha \in S_{\otimes}$ $(\boldsymbol{\alpha_0})$ (as illustrated in Fig. 1.9). Consequently, we can define the division region $S_{\oslash}(\alpha)$ of $\boldsymbol{\alpha}$ as the shadow region of Fig. 1.8. Furthermore, $S_{\otimes}$ $(\boldsymbol{\alpha}) \subseteq S_{\otimes}$ $(\boldsymbol{\alpha_0})$if and only if α is an element of $\alpha \in S_{\otimes}$ $(\boldsymbol{\alpha_0})$.

In the above (1)-(4), we have analyzed some properties of the basic operations between IFNs, including "⊕", "⊖", "⊗" and "⊘", As know, the

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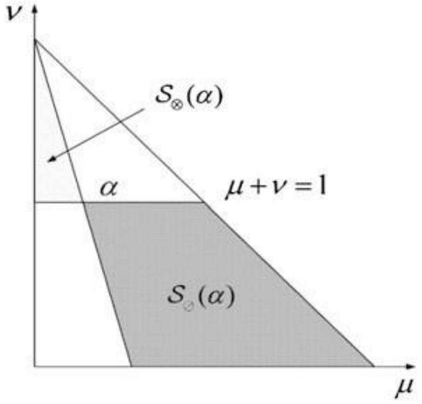

Fig. 1.8 Division region $S_{\oslash}(\boldsymbol{\alpha})$ of $\boldsymbol{\alpha}$

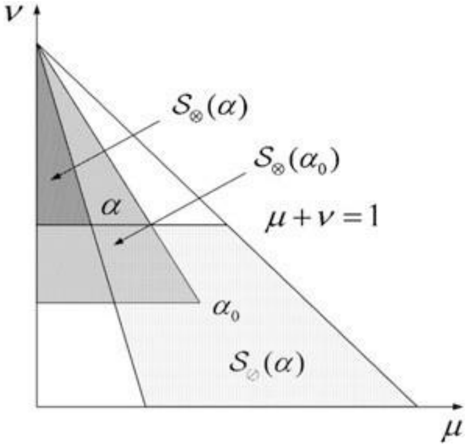

Fig. 1.9 Multiplication region of $\boldsymbol{\alpha_0}$ in a division region

scalar-multiplication and the power operations of IFNs are essentially the addition and the multiplication of IFNs, respectively, and some detailed analyses (Lei and Xu 2016a) can be processed as follows:

Firstly, we introduce two symbols $S\lambda_{\alpha}$ and $S_{\alpha}\lambda$, which represent the set $\{\beta|\beta = \lambda\alpha, \lambda \in (0, \infty)\}$ and $\{\beta|\beta = \alpha^{\lambda}, \lambda \in (0, \infty)\}$, respectively. For any given IFN $\boldsymbol{\alpha_0} = (\mu_0, v_0)$, we can get the following conclusions after analyzing the mathematical expression of $\lambda\boldsymbol{\alpha_0}$ **:**

(1) $\lambda\boldsymbol{\alpha_0}$ can be considered as a function of the variable λ, and the value of $\lambda\boldsymbol{\alpha_0}$ will depend on the parameter λ that varies from zero to the positive infinity.

(2) When $\lambda\boldsymbol{\alpha_0} = (\mu, v)$**,** we can calculate λ if only $\mu_0 \neq 0$, $\mu_0 \neq 1$, $v_0 \neq 0$ and $v_0 \neq 1$

(3) The image of $\lambda\boldsymbol{\alpha_0}$ can be represented as a function $v(\mu)$ in the $\mu - v$ plane, whose mathematical expression is

$$v(\mu) = v_0^{\frac{\ln(1-\mu)}{\ln(1-\mu_0)}}$$

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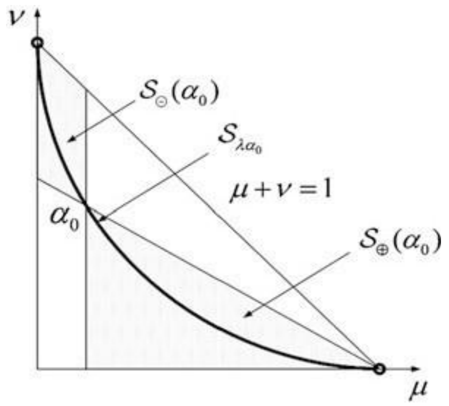

Fig. 1.10 The image of the scalar-multiplication $S_{\lambda\alpha_0}$of $\boldsymbol{\alpha_0}$

(4) $\lambda\boldsymbol{\alpha_0}$ can also be understated as a function $\mu(v)$ in the $\mu - v$ plane, where

$$\mu(v) = 1 - (1-\mu_0)^{\frac{\ln v}{\ln v_0}}$$

Next, we provide some analyses about the function $v(\mu)$, and $\mu(v)$ can also be analyzed in a similar way.

(1) $v(\mu)$ satisfies $v(\mu_0) = v_0$, which indicates $1(\mu_0, v_0) = (\mu_0, v_0)$ when the parameter λ =1.

(2) $v(1) = 0$ reveals that $\lambda\boldsymbol{\alpha_0} \to (1,0)$ when $\lambda \to +\infty$.

(3) $v(0) = 1$ represents that $\lambda\boldsymbol{\alpha_0} \to (0,1)$ if $\lambda \to 0$.

(4) Because $\lambda\alpha_0 = \alpha_0 \oplus (\lambda-1)\,\boldsymbol{\alpha_0}(\lambda > 1)$, there must be $\lambda\alpha_0 \in \boldsymbol{S_{\oplus}(\alpha_0)}$.

(5) When $0 < \lambda < 1$, there exists $\lambda\alpha_0 \in \boldsymbol{S_{\ominus}(\alpha_0)}$ due to $\lambda\alpha_0 = \alpha_0 \ominus (1-\lambda)\,\boldsymbol{\alpha_0}$.

Based on the above (1)-(5), the images of the scalar-multiplication and the power operation of IFNs can be shown in Fig. 1.10 (Lei and Xu 2016a) and Fig. 1.11 (Lei and Xu 2016a), respectively.

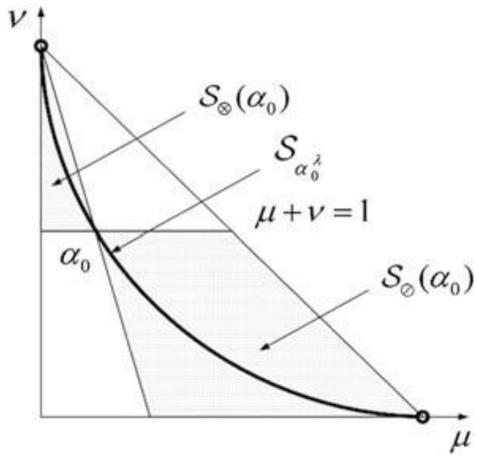

Fig. 1.11 The image of the power operation $S_{\alpha_0^{\lambda}}$ of $\boldsymbol{\alpha_0}$

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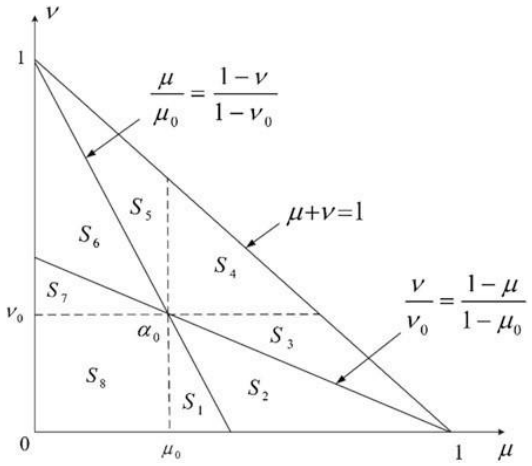

Fig. 1.12 The change region and the non-change region of $\boldsymbol{\alpha_0}$

Up to this point, we can address the question posed at the start of this section, specifically whether there is an IFN β that satisfies the equation $\boldsymbol{\alpha} = \boldsymbol{\alpha_0} * \boldsymbol{\beta}$, where "*" represents one of the four fundamental operations ("⊕", "⊖", "⊗", and "⊘"), for any two specified IFNs $\boldsymbol{\alpha_0}$ and $\boldsymbol{\alpha}$. The images depicting the operations of IFNs clearly indicate a negative response. To enhance this situation, we present a new concept regarding the change region of $\boldsymbol{\alpha_0}$ (Lei and Xu 2015b), defined as the set $\{\alpha_0 * x | x \in ▲\}$.The representation can be illustrated by the area $S_1 \cup S_2 \cup S_3 \cup S_5 \cup S_6 \cup S_7$ as shown in Fig. 1.12 (Lei and Xu 2015b). The area defined by $S_4 \cup S_8$ (Lei and Xu 2015b) is referred to as the non-change region of$\boldsymbol{\alpha_0}$. Consequently, the following conclusions can be drawn (Lei and Xu 2015b):

(1) For any IFN, there exists an IFN $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha} = \boldsymbol{\alpha_0} * \boldsymbol{\beta}$, provided that α is within the change region of $\boldsymbol{\alpha_0}$**.**

(2) If α is situated within the non-change region of α_0, then it follows that there cannot exist an IFN β such that **α = α_0*β**.

1.2.2 Algebraic Analysis of the Operations of IFNs

The examination of the operations of IFNs through algebraic methods enables researchers to gain deeper insights into the intricate signaling pathways that govern their functionality. By deconstructing the interactions between these cytokines and their receptors into mathematical formulations, researchers can enhance their ability to forecast the behaviour of IFNs across various physiological scenarios. This method has been crucial in clarifying the primary elements that govern the efficacy of IFNs in immune responses and disease mechanisms. Further research in this area could lead to the identification of novel targets for therapeutic interventions focused on modulating IFN signaling pathways through algebraic analysis. Researchers have employed mathematical models to analyses the interactions between varying concentrations of IFNs and their receptors, aiming to elucidate the activation of specific immune responses. Through the adjustment of these variables within their

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equations, researchers are able to forecast the potential effects of enhancing or inhibiting specific pathways on the overall immune response to infections or diseases. The precision and control afforded by algebraic analysis may facilitate the creation of more targeted and effective treatments for conditions in which IFN signaling is crucial.

In this subsection, we will give some algebraic properties of IFNs.

Theorem 1.1 (Lei and Xu 2015c) Let $\boldsymbol{\alpha}$ be any IFN (μ_α, v_α) $\boldsymbol{O}$ be (0, 1), and $\boldsymbol{E}$ be (1,0). Then we have

(1) $\boldsymbol{\alpha} \oplus O =$ **;** $\boldsymbol{\alpha} \ominus O = \boldsymbol{\alpha}$; $\boldsymbol{\alpha} \ominus \boldsymbol{\alpha} = \boldsymbol{O}$; $\boldsymbol{\alpha} \oplus \boldsymbol{E} =$ **E**

(2) $\boldsymbol{\alpha} \otimes \boldsymbol{E} = \boldsymbol{\alpha}$; $\boldsymbol{\alpha} \oslash \boldsymbol{E} = \boldsymbol{\alpha}$; $\boldsymbol{\alpha} \oslash \boldsymbol{\alpha} = \boldsymbol{E}$; $\boldsymbol{\alpha} \otimes O = \boldsymbol{O}$

(3) When $\mu_\alpha \neq 1$ and $v_\alpha \neq 0$, the expression $0\boldsymbol{\alpha}$ is meaningful and equal to $\boldsymbol{O}$.

(4) If $\mu_\alpha \neq 0$ and $v_\alpha \neq 1$, then $\boldsymbol{\alpha}^0$ is meaningful and there is $\boldsymbol{\alpha}^0$ = **E.**

$Proof$ The conclusions (1) and (2) can be easily proven according to the operational laws of addition and multiplication between IFNs. Hence, their proofs are omitted here. Next, we analyze (3) and (4), Because 0^0 and 1-$(1-1)^0$ are both meaningless, we give a restriction on the parameter λ of the scalar-multiplication $\lambda\boldsymbol{\alpha}$ and the power operation α^λ of IFNs, that is $\lambda > 0$. However, in most cases, $\lambda\boldsymbol{\alpha}$ and α^λ allow that λ = 0 if only λ satisfies these conditions in (3) and (4). It is worth. pointing out that if there is no special instruction, $\boldsymbol{\alpha}$ of $0\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^0$ are respectively assumed to satisfy the conditions in (3) and (4) in this book.

From Theorem 1.1, we can get a fact that $\boldsymbol{O}$ and $\boldsymbol{E}$ are respectively similar to zero and unity in real number field to some extent.

Theorem 1.2 (Xu and Cai 2012; Lei and Xu 2015c) Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and γ be three IFNs, λ_1 and λ_2 be two real number meeting $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_1 \geq \lambda_2$. Then

(1) $\boldsymbol{\alpha} \oplus \boldsymbol{\beta} = \oplus\, \boldsymbol{\alpha}$; $\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = \boldsymbol{\beta} \otimes \boldsymbol{\alpha}$

(2) $(\boldsymbol{\alpha} \oplus \boldsymbol{\beta}) \oplus \boldsymbol{\gamma} = \boldsymbol{\alpha} \oplus (\boldsymbol{\beta} \oplus \boldsymbol{\gamma})$; $(\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \otimes \boldsymbol{\gamma} = \boldsymbol{\alpha} \otimes (\boldsymbol{\beta} \otimes \boldsymbol{\gamma})$

(3) $\lambda_1\,(\boldsymbol{\alpha} \oplus \boldsymbol{\beta}) = \lambda_1\,\boldsymbol{\alpha} \oplus \lambda_1\,\boldsymbol{\beta}$; $(\boldsymbol{\alpha} \otimes \boldsymbol{\beta})^{\boldsymbol{\lambda_1}} = \boldsymbol{\alpha}^{\boldsymbol{\lambda_1}} \otimes \boldsymbol{\beta}^{\boldsymbol{\lambda_1}}$

(4) $\lambda_1(\boldsymbol{\beta} \ominus \boldsymbol{\alpha}) = \lambda_1\,\boldsymbol{\beta} \ominus \lambda_1\,\boldsymbol{\alpha}$; $(\boldsymbol{\beta} \oslash \boldsymbol{\alpha})^{\boldsymbol{\lambda_1}} = \boldsymbol{\beta}^{\boldsymbol{\lambda_1}} \oslash \boldsymbol{\alpha}^{\boldsymbol{\lambda_1}}$

(5) $(\lambda_1 + \lambda_2)\,\boldsymbol{\alpha} = \lambda_1\,\boldsymbol{\alpha} \oplus \lambda_2\,\boldsymbol{\alpha}$; $\boldsymbol{\alpha}^{\boldsymbol{\lambda_1}+\boldsymbol{\lambda_2}} = \boldsymbol{\alpha}^{\boldsymbol{\lambda_1}} \otimes \boldsymbol{\alpha}^{\boldsymbol{\lambda_2}}$

(6) $(\lambda_1 - \lambda_2)\,\boldsymbol{\alpha} = \lambda_1\,\boldsymbol{\alpha} \ominus \lambda_2\boldsymbol{\alpha}$; $\boldsymbol{\alpha}^{\boldsymbol{\lambda_1}-\boldsymbol{\lambda_2}} = \boldsymbol{\alpha}^{\boldsymbol{\lambda_1}} \oslash \boldsymbol{\alpha}^{\boldsymbol{\lambda_2}}$

$Proof$ According to the addition and the multiplication between IFNs, it is easy to get (1), (2), (3) and (4), which shows actually the commutative law and the associative law of IFNs. Next, we will prove (3) as follows:

$$\lambda_1\,\boldsymbol{\alpha} \oplus \lambda_1\boldsymbol{\beta} = \left(1-(1-\mu_\alpha)^{\lambda_1}, \mu_\alpha^{\lambda_1}\right) \oplus \left(1-(1-\mu_\beta)^{\lambda_1}, \mu_\beta^{\lambda_1}\right)$$

$$= (1-(1-\mu_\alpha)^{\lambda_1}\ \ (1-\mu_\beta)^{\lambda_1}, \mu_\alpha^{\lambda_1}, \mu_\beta^{\lambda_1})$$

$$= (1-((1-\mu_\alpha)\,(1-\mu_\beta))^{\lambda_1}, (\mu_\alpha\mu_\beta)^{\lambda_1})$$

$$= \lambda_1(\boldsymbol{\alpha} \oplus \boldsymbol{\beta})$$

1 Basic Concepts Related to Intuitionistic Fuzzy Numbers

Similarly, we can get $(\boldsymbol{\alpha} \otimes \boldsymbol{\beta})^{\lambda_1} = \boldsymbol{\alpha}^{\boldsymbol{\lambda_1}} \otimes \boldsymbol{\beta}^{\boldsymbol{\lambda_1}}$**.** Considering the conclusion (4), we will prove it in two different cases (Case 1 and Case 2):

Case 1. Because $0 \le \frac{v_\beta}{v_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$, there is $0 \le \left(\frac{v_\beta}{v_\alpha}\right)^{\lambda_1} \le \left(\frac{1-\mu_\beta}{1-\mu_\alpha}\right)^{\lambda_1} \le 1$. Hence, we also have a fact that the result of $\lambda_1 \boldsymbol{\beta} \ominus \lambda_1 \boldsymbol{\alpha} = \left(1 - \frac{(1-\mu_\beta)^{\lambda_1}}{(1-\mu_\alpha)^{\lambda_1}}, \frac{v_\beta^{\lambda_1}}{v_\alpha^{\lambda_1}}\right)$ is still an IFN if only $\boldsymbol{\beta} \ominus \boldsymbol{\alpha} = \left(\frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right) = \left(1 - \frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right)$ is an IFN. Then, we will prove (4) when $\boldsymbol{\beta} \ominus \boldsymbol{\alpha} = \left(\frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right)$ is still an IFN, which means $0 \le \frac{v_\beta}{v_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$ holds, and we have the following process:

$$\lambda_1 \boldsymbol{\beta} \ominus \lambda_1 \boldsymbol{\alpha} = (1-(1-\mu_\beta)^{\lambda_1}, \mu_\beta^{\lambda_1}) \ominus (1-(1-\mu_\alpha)^{\lambda_1}, \mu_\alpha^{\lambda_1})$$

$$= \left(1 - \frac{(1-\mu_\beta)^{\lambda_1}}{(1-\mu_\alpha)^{\lambda_1}}, \frac{\mu_\beta^{\lambda_1}}{\mu_\alpha^{\lambda_1}}\right) = \left(1 - \left(\frac{1-\mu_\beta}{1-\mu_\alpha}\right)^{\boldsymbol{\lambda_1}}, \left(\frac{v_\beta}{v_\alpha}\right)^{\boldsymbol{\lambda_1}}\right)$$

On the other hand, there is also

$$\lambda_1 (\boldsymbol{\beta} \ominus \boldsymbol{\alpha}) = \lambda_1 \left(1 - \frac{1-\mu_\beta}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right) = \left(1 - \left(\frac{1-\mu_\beta}{1-\mu_\alpha}\right)^{\boldsymbol{\lambda_1}}, \left(\frac{v_\beta}{v_\alpha}\right)^{\boldsymbol{\lambda_1}}\right)$$

Hence, in this case, $\lambda_1 (\boldsymbol{\beta} \ominus \boldsymbol{\alpha}) = \lambda_1 \boldsymbol{\beta} \ominus \lambda_1 \boldsymbol{\alpha}$ holds.

Case 2. When $0 \le \frac{v_\beta}{v_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$does not hold, which means that any result of

$\boldsymbol{\beta} \ominus \boldsymbol{\alpha} = \left(\frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha}, \frac{v_\beta}{v_\alpha}\right)$ and $\lambda_1 \boldsymbol{\beta} \ominus \lambda_1 \boldsymbol{\alpha} = \left(1 - \frac{(1-\mu_\beta)^{\lambda_1}}{(1-\mu_\alpha)^{\lambda_1}}, \frac{v_\beta^{\lambda_1}}{v_\alpha^{\lambda_1}}\right)$ is not an IFN, we get $\boldsymbol{\beta} \ominus \boldsymbol{\alpha} = \boldsymbol{O}$ **and** $\lambda_1 \boldsymbol{\beta} \ominus \lambda_1 \boldsymbol{\alpha} = \boldsymbol{O}$ according to the definition of subtraction in Definition 1.5. Thus, $\lambda_1 \boldsymbol{\beta} \ominus \lambda_1 \boldsymbol{\alpha} = \boldsymbol{O} = \lambda_1 \boldsymbol{O} = \lambda_1 (\boldsymbol{\beta} \ominus \boldsymbol{\alpha})$

According to Case I and Case 2, we have $\lambda_1(\boldsymbol{\beta} \ominus \boldsymbol{\alpha}) = \lambda_1 \boldsymbol{\beta} \ominus \lambda_1 \boldsymbol{\alpha}$ holds. The equation $(\boldsymbol{\beta} \oslash \boldsymbol{\alpha})^{\boldsymbol{\lambda_1}} = \boldsymbol{\beta}^{\boldsymbol{\lambda_1}} \oslash \boldsymbol{\alpha}^{\boldsymbol{\lambda_1}}$ can be proven in the same manner.

In addition, based on the laws of basic operations of IFNs, (5) and (6) can be proven easily, which is omitted here.

Theorem 1.3 (Lei and Xu 2015c) If $\alpha_1 = (\mu_1, v_1)$, $\alpha_2 = (\mu_2, v_2)$ and $\alpha_3 = (\mu_3, v_3)$, which satisfy the condition $S_\oplus(\boldsymbol{\alpha}_1) \subseteq S_\oplus(\boldsymbol{\alpha}_2) \subseteq S_\oplus(\boldsymbol{\alpha}_3)$, then

(1) $(\boldsymbol{\alpha}_1 \oplus \boldsymbol{\alpha}_2) \ominus (\boldsymbol{\alpha}_2 \oplus \boldsymbol{\alpha}_3) = \boldsymbol{\alpha}_1 \ominus \boldsymbol{\alpha}_3$.

(2) $(\boldsymbol{\alpha}_1 \ominus \boldsymbol{\alpha}_3) \ominus (\boldsymbol{\alpha}_2 \ominus \boldsymbol{\alpha}_3) = \boldsymbol{\alpha}_1 \ominus \boldsymbol{\alpha}_2$.

(3) $(\boldsymbol{\alpha}_1 \ominus \boldsymbol{\alpha}_2) \oplus (\boldsymbol{\alpha}_2 \oplus \boldsymbol{\alpha}_3) = \boldsymbol{\alpha}_1 \ominus \boldsymbol{\alpha}_3$.

Proof Based on the operational laws IFNs, the equation of (1) can be calculated as:

$(\boldsymbol{\alpha}_1 \oplus \boldsymbol{\alpha}_2) \ominus (\boldsymbol{\alpha}_2 \oplus \boldsymbol{\alpha}_3) = (1-(1-\mu_1)(1-\mu_2), v_1 v_2) \ominus (1 -$
$(1-\mu_2)(1-\mu_3), v_2 v_3)$

$$= \left(\frac{(1-\mu_2)(1-\mu_3)-(1-\mu_1)(1-\mu_2)}{(1-\mu_2)(1-\mu_3)}, \frac{v_1 v_2}{v_2 v_3}\right)$$

$$= \left(1-\frac{1-\mu_1}{1-\mu_3}, \frac{v_1}{v_3}\right) = \left(\frac{\mu_1-\mu_3}{1-\mu_3}, \frac{v_1}{v_3}\right)$$

$$= \boldsymbol{\alpha}_1 \ominus \boldsymbol{\alpha}_3$$

In the same way, the proof of (2) can be processed as follows:

$$(\boldsymbol{\alpha}_1 \ominus \boldsymbol{\alpha}_3) \ominus (\boldsymbol{\alpha}_2 \ominus \boldsymbol{\alpha}_3) = \left(\frac{\mu_1-\mu_3}{1-\mu_3}, \frac{v_1}{v_3}\right) \ominus \left(\frac{\mu_2-\mu_3}{1-\mu_3}, \frac{v_2}{v_3}\right)$$

$$= \left(\frac{\frac{\mu_1-\mu_3}{1-\mu_3}-\frac{\mu_2-\mu_3}{1-\mu_3}}{1-\frac{\mu_2-\mu_3}{1-\mu_3}}, \frac{v_1}{v_3}\frac{v_3}{v_2}\right) = \left(\frac{\mu_1-\mu_2}{1-\mu_2}, \frac{v_1}{v_2}\right)$$

$$= \boldsymbol{\alpha}_1 \ominus \boldsymbol{\alpha}_2$$

Moreover, the equation of (3) can be proved as follows:

$$(\boldsymbol{\alpha}_1 \ominus \boldsymbol{\alpha}_2) \ominus (\boldsymbol{\alpha}_2 \ominus \boldsymbol{\alpha}_3) = \left(\frac{\mu_1-\mu_2}{1-\mu_2}, \frac{v_1}{v_3}\right) \oplus \left(\frac{\mu_2-\mu_3}{1-\mu_3}, \frac{v_2}{v_3}\right)$$

$$= \left(1-(1-\frac{\mu_1-\mu_2}{1-\mu_2}\right) = \left(1-\frac{\mu_2-\mu_3}{1-\mu_3}\right), \frac{v_1}{v_3}\frac{v_2}{v_3}\right) = \left(\frac{\mu_1-\mu_3}{1-\mu_3}, \frac{v_1}{v_3}\right)$$

$$= \boldsymbol{\alpha}_1 \ominus \boldsymbol{\alpha}_3$$

The proof of Theorem 1.3 is completed,

Theorem 1.4 (Lei and Xu 2015c) If $\boldsymbol{\alpha}_1 = (\mu_1, v_1)$, $\boldsymbol{\alpha}_2 = (\mu_2, v_2)$ and $\boldsymbol{\alpha}_3 = (\mu_3, v_3)$. which satisfy the condition $S_{\otimes}(\boldsymbol{\alpha}_1) \subseteq S_{\otimes}(\boldsymbol{\alpha}_2) \subseteq S_{\otimes}(\boldsymbol{\alpha}_3)$,

then we have

(1) $(\boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_2) \otimes (\boldsymbol{\alpha}_2 \oslash \boldsymbol{\alpha}_3) = \boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_3$.

(2) $(\boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_3) \oslash (\boldsymbol{\alpha}_2 \oslash \boldsymbol{\alpha}_3) = \boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_2$.

(3) $(\boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_2) \oslash (\boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_3) = \boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_3$.

$Proof$ Firstly, we prove (1) as follows

$$(\boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_2) \otimes (\boldsymbol{\alpha}_2 \oslash \boldsymbol{\alpha}_3) = \left(\frac{\mu_1}{\mu_2}, \frac{v_1-v_2}{1-v_2}\right) \otimes \left(\frac{\mu_2}{\mu_3}, \frac{v_2-v_3}{1-v_3}\right)$$

$$= \left(\frac{\mu_1}{\mu_3}, 1-\left(1-\frac{v_1-v_2}{1-v_2}\right)\left(1-\frac{v_2-v_3}{1-v_3}\right)\right) = \left(\frac{\mu_1}{\mu_3}, \frac{v_1-v_3}{1-v_3}\right)$$

$$= \boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_3$$

Next, we can prove the proof of (2):

$$(\boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_3) \oslash (\boldsymbol{\alpha}_2 \oslash \boldsymbol{\alpha}_3) = \left(\frac{\mu_1}{\mu_3}, \frac{v_1-v_3}{1-v_3}\right) \oslash \left(\frac{\mu_2}{\mu_3}, \frac{v_2-v_3}{1-v_3}\right)$$

$$= \left(\frac{\mu_1}{\mu_3}\frac{\mu_3}{\mu_2}, \frac{\frac{v_1-v_3}{1-v_3}-\frac{v_2-v_3}{1-v_3}}{1-\frac{v_2-v_3}{1-v_3}}\right) = \left(\frac{\mu_1}{\mu_2}, \frac{v_1-v_2}{1-v_2}\right)$$

$$= \boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_2$$

Finally, we prove the conclusion (3):

$$(\boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2) \oslash (\boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_3) = (\mu_1\mu_2, 1-(1-v_1)(1-v_2)) \oslash (\mu_2\mu_3, 1-(1-v_2)(1-v_3))$$

$$= \left(\frac{\mu_1\mu_2}{\mu_2\mu_3}, \frac{(1-v_2)(1-v_3)-(1-v_1)(1-v_2)}{(1-v_2)(1-v_3)}\right) = \left(\frac{\mu_1}{\mu_3}, \frac{v_1-v_3}{1-v_3}\right)$$

$$= \boldsymbol{\alpha}_1 \oslash \boldsymbol{\alpha}_3$$

which completes the proofs.

Conclusion:

The basic operations between Intuitionistic fuzzy numbers involve addition, subtraction, multiplication, and division. Geometrical analysis of these operations helps visualize how IFNs interact with each other and how their values change. Algebraic analysis, on the other hand, provides a more formal and mathematical understanding of how these operations can be performed and manipulated. By combining both approaches, a comprehensive understanding of the behavior of IFNs had been achieved. This understanding allows researchers to make informed decisions when using IFNs in various applications, such as decision-making processes and pattern recognition systems. Furthermore, the combination of geometrical and algebraic analyses helps in the development of efficient algorithms for manipulating IFNs and solving complex problems in a more effective manner. Overall, the study of IFNs through these two perspectives offers a holistic approach to analyzing and utilizing these unique mathematical entities.

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